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Critique of Dilley's N/D generation of the ρ resonance

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Rigorous sum rules for negative moments of the discontinuity across the left-hand cut of the $\pi\pi$ P wave are derived and analyzed. A model by Dilley wherein the ρ resonance emerges from elastic N/D equations is shown to be severely inconsistent with these sum rules. Dilley's method for selecting the input left cut is analyzed and shown to be strongly biased in favor of generating a ρ. Because of this bias, together with the aforementioned violation of sum rules, Dilley's model does not comprise evidence that the ρ is generated by forces in the $\pi\pi$ channel. Numerous successes of the quark model suggest otherwise.

I. INTRODUCTION AND SUMMARY

In an earlier Letter[1] (hereafter referred to as T1), rigorous sum rules were derived for the second and all higher negative moments of the discontinuity across the left cut of the $\pi\pi$ P wave. These moments were expressed strictly in terms of physical-region absorptive parts. Subsequent to the publication of T1, Dilley published an elastic N/D model[2] which generates a ρ in good agreement with experiment. This model was acknowledged to violate the sum rules of T1, but the violations were alleged to be minor and acceptable.

The dynamical origin of the ρ resonance is an issue of major importance, so we analyze here the model of Dilley. We shall find that his method for selecting a left cut is strongly biased in favor of generating a ρ. We shall also find the distant left cut of Dilley's model to be severely inconsistent with rigorous sum rules. The aforementioned bias, together with the violation of sum rules, leads us to conclude that Dilley's work should not be regarded as evidence that the ρ is generated by forces in the $\pi\pi$ channel. This conclusion is of course supported, albeit indirectly, by numerous successes of the quark model.[3]

To facilitate our analysis, we shall cast the sum rules of T1 into a new form which is better suited for studying the distant left cut. We shall also consider in some detail the general mechanism of resonance generation, before proceeding to our critique of Dilley's model.

II. DERIVATION OF SUM RULES

We denote the $\pi\pi$ elastic amplitudes by $A^I(\nu, \cos\theta)$, where I denotes the (direct) s-channel isospin, and $\nu \equiv |\vec{q}\,|^2 = \frac{1}{4}(s - 4m_\pi^2)$. We use units wherein $m_\pi = \hbar = c = 1$, and our normalization is such that the partial waves may be written as

$$A^{(l)I}(\nu) = \tfrac{1}{2} i(1 + 1/\nu)^{1/2}[1 - \eta_l^I \exp(2i\delta_l^I)] \ ,$$

where η_l^I denotes the elasticity $(0 \leq \eta_l^I \leq 1)$, and the phase shifts δ_l^I are real. Bose symmetry implies that $A^{(l)I}$ vanishes when $(l + I)$ is odd.

We shall denote the combination of amplitudes with isospin I in the t channel by

$$T^I(\nu, \cos\theta) = \sum_{I'=0}^{2} \beta_{II'} A^{I'}(\nu, \cos\theta) \ ,$$

where $\beta = \beta^{-1}$ denotes the s-t crossing matrix. The elements we shall need here are $\beta_{1I} = \frac{1}{3}, \frac{1}{2}$, and $-\frac{5}{6}$ for $I = 0$, 1, and 2, respectively.

The Froissart-Gribov representation implies for odd l that

$$A^{(l)1}(\nu) = \frac{4}{\pi\nu}\int_0^\infty d\nu' Q_l\left(1 + 2\frac{\nu'+1}{\nu}\right) \times \mathrm{Im}\,T^1\left(\nu', 1 + 2\frac{\nu+1}{\nu'}\right), \qquad (1)$$

which is valid for $\mathrm{Re}\,\nu < 0$. Although one could base sum rules on a comparison of Eq. (1) for $l = 1$ with the standard relation

$$A^{(1)1}(\nu) = \frac{\nu}{\pi}\left(\int_{-\infty}^{-1} d\nu' + \int_0^\infty d\nu'\right)\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu' - \nu)}, \qquad (2)$$

the resulting sum rules would all involve $\mathrm{Im}\,T^1$ away from the forward direction. The $\rho\pi\pi$ Regge residue function is now known[4] for $-1.0\ \mathrm{GeV}^2 < t < 0.1\ \mathrm{GeV}^2$, so such sum rules could readily be evaluated. When T1 was written, however, the $\rho\pi\pi$ residue was not known for $t \neq 0$, so an alternative procedure was followed. Since Dilley compared his model with the sum rules of T1, we follow here the procedure of T1, and write the P wave as

$$A^{(1)1}(\nu) = \tfrac{1}{3}[A^1(\nu, 1) - \tilde{A}^1(\nu)] \ , \qquad (3)$$

where

$$\tilde{A}^1(\nu) \equiv \sum_{l=3}^{\infty} (2l + 1) A^{(l)1}(\nu) \ .$$

The forward amplitude $A^1(\nu, 1)$ satisfies a well-known dispersion relation,[5] and $\tilde{A}^1(\nu)$ can be obtained from Eq. (1) together with the observations that

$$Q_l(z) = -\tfrac{1}{2}\int_{-1}^{1} dz'\,\frac{P_l(z')}{z'-z}\,,$$

$$\sum_{\substack{l=3\\ \text{odd}}}^{\infty}(2l+1)P_l(z) = \delta(z-1) - \delta(z+1) - 3P_1(z)\,.$$

Thus Eqs. (2) and (3) lead to

$$\int_{-\infty}^{-1} d\nu'\,\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)} = \tfrac{1}{3}\int_0^{\infty}\frac{d\nu'}{(\nu'+1)(\nu'+\nu+1)}\left[\mathrm{Im}T^1(\nu',1) + \frac{(\nu+1)(2\nu'+1)}{\nu'(\nu'-\nu)}\mathrm{Im}A^1(\nu',1)\right]$$
$$-\tfrac{1}{3}\int_0^{\infty} d\nu'\left[\frac{1}{(\nu'+1)(\nu'+\nu+1)} - \frac{12}{\nu^2}Q_1\left(1+2\frac{\nu'+1}{\nu}\right)\right]\mathrm{Im}T^1\left(\nu', 1+2\frac{\nu+1}{\nu'}\right)$$
$$-\int_0^{\infty} d\nu'\,\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)}\,. \tag{4}$$

Since the left-hand side of Eq. (4) is manifestly analytic for $|\nu|<1$, both sides can be expanded as a power series in ν, with equal coefficients. In this way, we obtain sum rules of the form

$$\int_{-\infty}^{-1} d\nu\,\frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^n} = \int_0^{\infty} d\nu\, B_n(\nu) \tag{5}$$

for all integers $n \geq 2$, where the functions $B_n(\nu)$ are defined *implicitly* by Eqs. (4) and (5). Note that $\tilde{A}^1(\nu)$ vanishes like ν^3 as $\nu \to 0$, so that B_2 and B_3 are independent of $\mathrm{Im}T^1$ *away* from the forward direction; this was the original reason for using Eq. (3).

For $-9<\nu<-1$, analyticity and crossing symmetry imply[6]

$$\mathrm{Im}A^{(1)1}(\nu) = \frac{2}{\nu}\int_0^{-\nu-1} d\nu' P_1\left(1+2\frac{\nu'+1}{\nu}\right)\sum_{I=0}^{2}\beta_{1I}\sum_{l=0}^{\infty}(2l+1)\mathrm{Im}A^{(l)I}(\nu')P_l\left(1+2\frac{\nu+1}{\nu'}\right). \tag{6}$$

Unfortunately, the Legendre series on the right-hand side of Eq. (6) diverges over part of the range of integration when $\nu<-9$.[6] It is this divergence which had previously prevented one from obtaining reliable information about $\mathrm{Im}A^{(1)1}$ for $\nu<-9$.

In T1, the sum rules were written in the form of Eq. (5) (up to overall multiplicative constants). The functions B_n were given explicitly for $n=2$, 3, and 4. Upon using Eq. (6) for the interval $-9<\nu<-1$, one finds that S- and P-wave contributions dominate both sides of Eq. (5), and that major cancellations occur between the two sides. [These cancellations are the reason why Dilley's model *almost* satisfies the sum rules (5).] These cancellations are no accident: it is readily proven that if one were to use Eq. (6) over the *entire* left cut, then *any* $\mathrm{Im}A^{(l)I}$ would make *identical* contributions to both sides of Eq. (5), for all $n \geq (l+2)$.[7]

Since our goal is to obtain information about $\mathrm{Im}A^{(1)1}$ for $\nu<-9$, let us use Eq. (6) for $-9<\nu<-1$, and cast the sum rules into the form

$$\int_{-\infty}^{-9} d\nu\,\frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^n} = \sum_{I=0}^{2}\beta_{1I}\sum_{l=0}^{\infty}\int_0^{8} d\nu\, F_n^{(l)}(\nu)\mathrm{Im}A^{(l)I}(\nu) + \int_8^{\infty} d\nu\, G_n(\nu)\,, \tag{7}$$

which defines *implicitly* the functions $F_n^{(l)}$ and G_n. Explicit expressions are given in Appendix A for $l=0$, 1, and 2, $n=2$, 3, 4, and 5. In practice, the $\mathrm{Im}A^{(l)I}$ with $l \geq 3$ are negligible for $0<\nu<8$ ($E_{\text{c.m.}}<840$ MeV), so that one has no need for the $F_n^{(l)}$ with $l \geq 3$.[8]

The information contained in the sum rules (7) is quite substantial, since

$$\frac{\nu}{\pi}\int_{-\infty}^{-9} d\nu'\,\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)} = \sum_{n=2}^{\infty}\frac{\nu^{n-1}}{\pi}\int_{-\infty}^{-9} d\nu'\,\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'^n} \tag{8}$$

for $|\nu|<9$. Thus the contribution of $\mathrm{Im}A^{(1)1}$ from $-\infty<\nu<-9$ to $A^{(1)1}$ throughout the circle $|\nu|<9$ is given by the sum rules (7), strictly in terms of physical-region absorptive parts.[9]

III. EVALUATION OF SUM RULES

In our evaluation of the right-hand sides of the sum rules (7), we divide the physical region into two parts: the region below 1.9 GeV, where experimental phase shifts are available, and the "high-energy" region above 1.9 GeV.

Inspection of the functions G_n given in Appendix A reveals that knowledge of $\mathrm{Im}A^1$, $\mathrm{Im}T^1$, and $\mathrm{Im}A^{(1)1}$ is required. Above 1.9 GeV, we use the fact that

TABLE I. R_k denotes contribution to right-hand side of sum rule (7) for $n=k$. The subscripts HE denote the high-energy region above 1.9 GeV. Input absorptive parts are described in Appendix B, together with their uncertainties. The four "total" values in the bottom row of this table are estimated to be uncertain by 29%, 33%, 18%, and 15%, respectively.

	10^2R_2	10^3R_3	10^4R_4	10^5R_5
S_0	0.46	−0.53	0.49	−0.45
S_2	−0.43	0.29	−0.21	0.16
P	−0.89	0.74	−1.91	2.14
D_0	0.96	−0.45	0.19	−0.19
D_2	−0.03	0.02	−0.01	0.01
F	0.30	0.25	0.03	0.02
$\mathrm{Im}\,T^0_{\mathrm{HE}}$	0.07	0.63	0.00	0.01
$\mathrm{Im}\,T^1_{\mathrm{HE}}$	1.78	0.00	0.01	−0.01
$\mathrm{Im}\,A^{(1)1}_{\mathrm{HE}}$	−0.54	−0.06	−0.01	0.00
Total	1.68	0.89	−1.42	1.69

$$A^1=\sum_{I=0}^{2}\beta_{1I}T^I,$$

and use Regge theory for the $\mathrm{Im}\,T^I$. Details of the absorptive parts used throughout this paper are given in Appendix B.

In Table I, the various contributions to the right-hand side of Eq. (7) are itemized for $n=2$, 3, 4, and 5. Observe that, for $n=2$, the sum rule is dominated by the contribution of $\mathrm{Im}\,T^1_{\mathrm{HE}}$. For $n=3$, several contributions are of comparable size. For $n=4$ and 5 (and all higher n), however, the sum rule is strongly dominated by low-energy contributions, primarily the ρ (P-wave) contribution. Hence for all $n\geq 4$, the right-hand side of Eq. (7) can be evaluated with nearly the same precision as one's knowledge of the ρ resonance parameters.

There is no mystery in the fact that high-order sum rules are dominated by low-energy absorptive parts. The functions $G_n(\nu)$ in Eq. (7) are such that the coefficient of each amplitude tends to zero for large ν at least as rapidly as ν^{-n}. In practice, one finds that every sum rule with $n\geq 4$ is strongly dominated by contributions from below 1 GeV, primarily the ρ contribution.

IV. DYNAMICS OF RESONANCE GENERATION

To understand what type of left cut would generate a resonance, it is useful to regard $A^{(1)1}$ as a sum of the two parts

$$A^{(1)1}(\nu)=A_{\mathrm{L}}(\nu)+A_{\mathrm{R}}(\nu),$$

where A_{L} and A_{R} denote the contributions from the left and right cuts, respectively, in the partial-wave dispersion relation (2).

The physical region corresponds to ν real and positive, where unitarity implies that

$$|\mathrm{Re}A^{(1)1}|\leq\tfrac{1}{2}\eta_1^1(1+1/\nu)^{1/2}. \quad (9)$$

Now suppose that

$$A_{\mathrm{L}}(\nu)>\tfrac{1}{2}\eta_1^1(1+1/\nu)^{1/2} \quad (10)$$

over some interval of positive ν. The unitarity constraint (9) then requires that A_{R} be negative. Since $\mathrm{Im}A^{(1)1}$ is positive for $\nu>0$, it follows that $A_{\mathrm{R}}(\nu)$ receives a *negative* contribution from the interval $0<\nu'<\nu$, and a *positive* contribution from the interval $\nu<\nu'<\infty$. In order for the inequality (9) to hold in the face of (10), it is necessary that A_{R} receive a larger contribution from the interval $0<\nu'<\nu$ than from the interval $\nu<\nu'<\infty$. *A resonance peak in* $\mathrm{Im}A^{(1)1}$ *at some* $\nu'<\nu$ *is ideally suited for this purpose*. In contrast, a monotonic growth of $\mathrm{Im}A^{(1)1}$ from threshold to a region substantially above ν would *not* render $A_{\mathrm{R}}(\nu)$ negative, and hence would not preserve unitarity.

The preceding remarks are well illustrated by the model of Kang and Lee,[10] who published sufficient information for one to compute their A_{L}. Kang and Lee represent the left cut by a few poles, and determine the pole parameters by requiring the solution of elastic N/D equations to contain a ρ resonance and to have the scattering length $a_1=0.038$ predicted by Weinberg.[11] Upon computing their A_{L}, one finds that A_{L} is *negative* between threshold and 850 MeV, but becomes positive and grows monotonically to exceed 0.55 above 1.4 GeV and to exceed 1.1 above 2.0 GeV. Hence unitarity demands that $A_{\mathrm{R}}<0$ above 1.4 GeV and that $A_{\mathrm{R}}<-0.6$ above 2.0 GeV. This requires a resonance below 1.4 GeV, and the details of the model are such that a broad ρ ($\Gamma>300$ MeV) occurs at 770 MeV.

The model of Kang and Lee is severely inconsistent with the sum rules (5) [and (7)], as has been discussed previously.[12] The point I wish to make here is that the model of Kang and Lee generates a ρ *not* because A_{L} is large in the ρ region (it vanishes at 850 MeV), but rather because A_{L} grows large *at substantially higher energies*.

In connection with the preceding remarks, it is well to remember that N/D phase shifts (without Castillejo-Dalitz-Dyson or bound-state poles) tend asymptotically to zero (Levinson's theorem). Hence any phase shift which rises through 90° will eventually fall back through 90°, giving rise to a *second* peak in $\mathrm{Im}(N/D)$. This second peak has no physical analog, since strong-interaction phase shifts do not fall back to zero. To whatever extent this second peak in $\mathrm{Im}(N/D)$ comprises an important singularity, the physical interpretation of an N/D calculation is problematic. In the model of Kang and Lee, the two peaks are closely spaced and the dip between them is shallow. Hence the two peaks are nearly merged into one, so that $\mathrm{Im}(N/D)$ bears an approximate resemblance to the physical $\mathrm{Im}A^{(1)1}$. At the same time, $\mathrm{Re}(N/D)$ has an unphysical behavior just above the resonance. We shall later describe an N/D model for $A^{(1)1}$ in which the two peaks are quite distinct and widely separated, but we shall find the *second* peak to play the dominant role in preserving unitarity over most of the region where the inequality (10) is satisfied. The model to be described has a ρ resonance in excellent agreement with experiment, but is quite unphysical because of the crucial role played by the second peak.

With regard to the relative importance of different order moments of the distant left cut, Eq. (8) indicates that only the low-order moments give appreciable contributions to A_L in the low-energy region. *Resonance generation depends critically, however, on the behavior of A_L at energies well above the resonance.* The *shape* of any resonance also depends critically on the behavior of A_L at these higher energies. *It is therefore essential that high-order moments have plausible values before one can claim a successful generation of the ρ.* Reliable estimates for high-order moments may easily be computed, for we have seen that high-order ($n \geq 4$) moments are determined to good precision by low-energy data (primarily by the ρ resonance).

V. THE MODEL OF DILLEY

Dilley has developed and used a method introduced by Balazs,[13] wherein one determines the distant left cut of $A^{(1)1}$ by requiring agreement for $-1 \leq \nu < 0$ between N/D and A_{FG} [where A_{FG} denotes the Froissart-Gribov $A^{(1)1}$ given by Eq. (1)]. A_{FG} is evaluated in terms of experimentally based absorptive parts, with the goal of obtaining the physically correct $A^{(1)1}$. The bias inherent to this method for selecting a distant left cut will be made clear in the following discussion.

Let us suppose that physically correct absorptive parts have been used in the evaluation of A_{FG}, so that A_{FG} agrees precisely (in its domain of validity) with the physical $A^{(1)1}$. Suppose also that D has no zeros on the physical sheet (i.e., no ghost or bound-state poles in N/D), so that N/D has the same domain of analyticity as $A^{(1)1}$ (i.e., only right and left cuts). *It is then rigorously true* (by analytic continuation) *that agreement between A_{FG} and N/D for $-1 \leq \nu < 0$ implies agreement between N/D and the physical $A^{(1)1}$ over the entire complex ν plane.* In particular, N/D and $A^{(1)1}$ must have identical discontinuities across their respective cuts, so that N/D *must* contain a ρ resonance identical to the physical ρ.

The preceding paragraph suggests that the gap-matching method is strongly biased toward generating an output ρ, provided only that the input absorptive parts used in evaluating A_{FG} have led to a good approximation for the physical $A^{(1)1}$. In the following paragraphs, we shall analyze this bias in quantitative detail.

Let us decompose A_L into a sum of terms A_{DL} and A_{NL} coming from the distant and nearby parts of the left cut, respectively:

$$A_L(\nu) = \frac{\nu}{\pi}\left(\int_{-\infty}^{-9} d\nu' + \int_{-9}^{-1} d\nu'\right) \frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)}$$
$$\equiv A_{DL}(\nu) + A_{NL}(\nu).$$

We are concerned here with the interval $-1 \leq \nu < 0$, over which the power series (8) is valid for A_{DL}. Using the moments given in Table I, we have

$$A_{DL}(\nu) \cong 5.32\times10^{-3}\nu + 2.8\times10^{-4}\nu^2 \tag{11}$$

within 1% for $|\nu| \leq 1$.

The function A_{NL} is readily computed, since Eq. (6) yields $\mathrm{Im}A^{(1)1}$ for $-9<\nu<-1$. The result turns out to be quite small because of a cancellation: $\mathrm{Im}A^{(1)1}$ is negative for $-9 \leq \nu < -5.86$, but positive for $-5.86 < \nu < -1$. A power series for A_{NL} converges for $|\nu|<1$, but the convergence is rather slow. We find that

$$A_{NL}(\nu) \cong 9\times10^{-5}\nu - 1.27\times10^{-3}\nu^2 + 7.5\times10^{-4}\nu^3 - 4.0\times10^{-4}\nu^4 \tag{12}$$

within 3% for $|\nu| \leq 0.5$, and within 26% for $|\nu| \leq 1$ [Eq. (12) yields $A_{NL}(-1) = -2.51\times10^{-3}$, while the true value is -3.39×10^{-3}].

Next we consider the term A_R arising from the right cut. Although A_R has a branch point at $\nu=0$, the near part of the cut is weak, and A_R can be approximated near threshold by a polynomial:

$$A_R(\nu) \cong 3.35\times10^{-2}\nu + 4.9\times10^{-3}\nu^2 \tag{13}$$

within 0.7% for $-1 \leq \nu < 0$. A_R is of course strongly dominated by the ρ contribution. We note in passing that Eqs. (11), (12), and (13) yield a value of 0.039 for the P-wave scattering length, in good agreement with Weinberg's current-algebra prediction.[11]

Since the ratio N/D has the same analytic structure as $A^{(1)1}$, it can be regarded as a sum of three terms analogous to our decomposition of $A^{(1)1}$:

$$N/D = A_{DL} + A_{NL} + A_R, \tag{14}$$

where the input A_{NL} is typically chosen on the basis of Eq. (6), and the input A_{DL} is chosen by some other method. The gap-matching method consists of varying A_{DL} until A_{DL} and the output A_R satisfy

$$A_{DL}+A_R=A_{FG}-A_{NL}. \quad (15)$$

The *crucial point* of this section is that the physical A_R is *very much larger* (six times larger) than the physical A_{DL} for $-1 \le \nu < 0$. Hence in the gap-matching method, *it is much more important for N/D to have an output ρ than for the distant left cut to be correct.*

To illustrate this bias in the gap-matching method, we consider the following approximation in the gap for the resonance-dominated $A^{(1)1}$:

$$A^{(1)1}(\nu) \cong A_R(\nu) \quad (16a)$$

$$\cong \left(\frac{m_\rho \Gamma_\rho}{4}\right) \frac{\nu}{\nu_\rho(\nu-\nu_\rho)} \quad (16b)$$

with $m_\rho = 5.07$ (i.e., 770 MeV) and $\Gamma_\rho = 1.09$ (i.e., 150 MeV). Proceeding with our illustration, we use a one-pole approximation for the left cut of N, and vary the pole position and residue until agreement is maximized between N/D and the $A^{(1)1}$ of Eq. (16b), for $-1 \le \nu \le 0$. If we interpret "maximum agreement" in terms of minimizing the integral

$$\Delta^2 \equiv \int_{-1}^{0} d\nu \left[\frac{N/D - A^{(1)1}}{A^{(1)1}}\right]^2,$$

then the pole in N is uniquely determined, and Δ_{min}^2 has the quite satisfactory value $\Delta_{min}^2 = 8.2 \times 10^{-6}$. The resulting N/D has an *excellent* output ρ, with $m_\rho = 768$ MeV and $\Gamma_\rho = 155$ MeV. The phase shift even reaches a maximum value of 163° near 2 GeV, before beginning a slow descent back through 90° down to zero at $\nu = \infty$. The calculation is highly successful in producing a ρ in agreement with experiment, but *this cannot be regarded as evidence that the ρ in Eq. (16b) is generated by exchange forces, because the amplitude* (16b) *has no left-hand cut.* The "success" of this N/D calculation is mearly evidence that the gap-matching method is strongly biased in favor of generating a ρ, *regardless* of whether the ρ in the amplitude being matched is generated by forces in the $\pi\pi$ channel.

Details of the preceding N/D calculation are rather interesting. The pole in N corresponds[14] to $\text{Im}N = a\delta(\nu - \bar{\nu})$, with $a = 5.28 \times 10^{12}$ and $\bar{\nu} = -7.12 \times 10^6$. The resulting left-hand cut in N/D is given by $\text{Im}(N/D) = b\delta(\nu - \bar{\nu})$, with $b = 7.02 \times 10^7$. Since N/D satisfies a dispersion relation identical in form to Eq. (2), we conclude that

$$(N/D)_L = \frac{b\nu}{\pi\bar{\nu}(\bar{\nu}-\nu)} \quad (17a)$$

$$\cong 4.4 \times 10^{-7}\nu, \quad (17b)$$

where the approximation (17b) is valid for $|\nu| \lesssim 10^5$. It follows that $(N/D)_L$ is less than 0.01 for $M_{\pi\pi} < 41$ GeV. Hence $(N/D)_L$ is *miniscule* throughout the range of center-of-mass energies spanned by existing accelerators. For $\nu > 1.35 \times 10^6$ (i.e., $M_{\pi\pi} > 320$ GeV), however, the inequality (10) is satisfied, and the discussion of Sec. IV implies that N/D must have a large right-hand cut over some region below 320 GeV. In addition to the ρ peak at 768 MeV, we find a second peak of enormous width, corresponding to a slow descent of the phase shift back down through 90°. This second peak has half-maxima at 160 GeV and 3500 GeV, with the 90° point at 740 GeV. It is clearly this second peak in $\text{Im}(N/D)$ which plays the dominant role in preserving unitarity throughout most of the region where the inequality (10) is satisfied. More specifically, the vanishing of N/D at infinity implies that

$$\lim_{\nu \to \infty} (N/D)_L = \frac{1}{\pi} \int_0^\infty d\nu \frac{\text{Im}(N/D)}{\nu}. \quad (18)$$

Equation (17a) implies a value of 3.14 for the left-hand side of Eq. (18), while the ρ peak contributes only 0.3 to the right-hand side of Eq. (18). The balance of 2.8 comes from the second, unphysical peak in $\text{Im}(N/D)$. Evidently this N/D solution bears no fundamental relationship to the physical $\pi\pi$ P wave, despite the beautiful ρ resonance contained within it.

Having displayed the bias inherent to the gap-matching method, we turn now to a comparison of Dilley's model with the sum rules (7). In practice, Dilley used an input ρ with a width of 115 MeV, and his favored solution has an output ρ with a width of 168 MeV. Since his input ρ and output ρ have nearly the same mass, the 46% discrepancy in widths should be reflected in large *percentage* discrepancies between the left- and right-hand sides of the sum rules (7). Table II displays the contributions of Dilley's input absorptive parts to these sum rules for $n = 2$, 3, and 4 (Dilley does not provide enough information in Ref. 2 for one to evaluate the sum rules with $n \ge 5$). For $n = 2$, the left- and right-hand sides disagree by a factor

TABLE II. Contributions to left- and right-hand sides of sum rules (7) in Dilley's model. Numerical data have been deduced from Table V of Ref. 2.

n	Left-hand side	Right-hand side
2	0.017	0.035
3	−0.0008	−0.0002
4	0.00005	−0.00010

of 2. For $n=3$, the left- and right-hand sides disagree by a factor of 4. For $n=4$, the left- and right-hand sides disagree in sign. These large violations of the sum rules (7) imply the Dilley's model for the distant left cut is highly unphysical, and tend to confirm our conjecture that Dilley's selection procedure obtains agreement between N/D and A_{FG} *by producing an output* ρ with reasonable mass and width, *at the cost* of large errors in the distant left cut.

We noted earlier that A_{FG} must agree with the physical $A^{(1)1}$ in order for the gap-matching method to assure an output ρ in N/D. This explains the fact that Dilley's "best" results were obtained when he modified his result for A_{FG} with correction terms which improved crossing symmetry, before requiring agreement of N/D with A_{FG}. This also explains why, in earlier work,[15] Gibbons and Dilley only obtained an output ρ when high-mass resonances (f_0 and g) were included in the computation of A_{FG}. Through duality, these resonances mimic Reggeized ρ exchange, which is required by crossing symmetry.

The input absorptive parts used in this paper lead to agreement between A_{FG} and $(A_L + A_R)$ within 1.3% for $-1 \leq \nu < 0$, when A_{DL} is computed from Eqs. (7) and (8), and A_{NL} is computed from Eq. (6). Hence the absorptive parts used here are highly consistent with crossing symmetry.

It is conceivable that if one were to include a large number of adjustable parameters in a model for the left cut, then one could satisfy some *smaller* number of sum rules and still have enough adjustable parameters to obtain an output ρ in reasonable agreement with experiment. Even if this were done, however, it would merely show that a finite number of constraints leaves freedom which permits *but does not imply* a generation of the ρ.

In order for a generation of the ρ to be *implied* by analyticity, unitarity, and crossing symmetry, one would have to include *only* as many degrees of freedom in the left cut as are constrained by sum rules (and/or by other unbiased conditions), and then generate a ρ for plausible values of the input absorptive parts. This has never been done. Even if it were achieved at some future time, the interpretation would be problematic, because of the fact stressed earlier that N/D phase shifts tend asymptotically to zero, giving rise to a second, unphysical peak in $\mathrm{Im}(N/D)$.

VI. CONCLUSIONS

We have seen that the gap-matching method is strongly biased in favor of generating a ρ, and that Dilley's model is severely inconsistent with the sum rules (7). Hence Dilley's model does not comprise evidence that the ρ is generated by forces in the $\pi\pi$ channel. Furthermore, numerous successes of the quark model suggest that low-lying resonances like the ρ are primarily diquark systems,[3] rather than dimeson systems. Hence it seems unlikely that the ρ resonance is generated by forces in the $\pi\pi$ channel.

APPENDIX A

Comparison of Eqs. (4), (6), and (7) leads to the following formulas:

$$F_2^{(0)} = \frac{4\nu - 23}{2187}, \quad F_3^{(0)} = \frac{5-\nu}{6561}, \quad F_4^{(0)} = \frac{8\nu - 37}{590490}, \quad F_5^{(0)} = \frac{44 - 10\nu}{7971615},$$

$$F_2^{(1)} = \frac{1}{729}\left(4\nu - 123 + \frac{818}{\nu}\right) - \frac{2}{\nu(\nu+1)}, \quad F_3^{(1)} = \frac{1}{2187}\left(-\nu + 27 - \frac{128}{\nu}\right),$$

$$F_4^{(1)} = \frac{1}{196830}\left(8\nu - 201 + \frac{826}{\nu}\right), \quad F_5^{(1)} = \frac{1}{2657205}\left(-10\nu + 240 - \frac{911}{\nu}\right),$$

$$F_2^{(2)} = \frac{10}{729}\left(\frac{2\nu}{3} - \frac{323}{6} + \frac{1277}{\nu} + \frac{1813}{\nu^2}\right) - \frac{60}{\nu^2}\ln\left(\frac{\nu+1}{9}\right) - \frac{10}{(\nu+1)^3}\left(12\nu + 37 + \frac{38}{\nu} + \frac{13}{\nu^2}\right),$$

$$F_3^{(2)} = \frac{10}{2187}\left(-\frac{\nu}{6} + \frac{71}{6} - \frac{203}{\nu} + \frac{1163}{\nu^2}\right) - \frac{10}{\nu^2(\nu+1)},$$

$$F_4^{(2)} = \frac{10}{196830}\left(\frac{4\nu}{3} - \frac{529}{6} + \frac{1321}{\nu} - \frac{5347}{\nu^2}\right),$$

$$F_5^{(2)} = \frac{10}{531441}\left(-\frac{\nu}{3} + \frac{316}{15} - \frac{2929}{10\nu} + \frac{1024}{\nu^2}\right),$$

$$G_2 = \frac{1}{3(\nu+1)^2}\,\mathrm{Im}\left[T^1(\nu,1) + \frac{2\nu+1}{\nu^2}A^1(\nu,1)\right] - \frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^2},$$

$$G_3 = \frac{1}{3(\nu+1)^3}\,\mathrm{Im}\left[-T^1(\nu,1) + \frac{\nu^3+(\nu+1)^3}{\nu^3}\,A^1(\nu,1)\right] - \frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^3}\ ,$$

$$G_4 = \frac{1}{3(\nu+1)^4}\,\mathrm{Im}\left[T^1(\nu,1) + \frac{(2\nu+1)(2\nu^2+2\nu+1)}{\nu^4}\,A^1(\nu,1) - \tfrac{1}{10}\,T^1(\nu,1+2/\nu)\right] - \frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^4}\ ,$$

$$G_5 = \frac{1}{3(\nu+1)^5}\,\mathrm{Im}\left[-T^1(\nu,1) + \frac{(2\nu+1)(\nu^4+2\nu^3+4\nu^2+3\nu+1)}{\nu^5}\,A^1(\nu,1) + \tfrac{1}{5}\,T^1(\nu,1+2/\nu) - \frac{\nu+1}{5\nu}\right.$$
$$\left.\times\,\frac{\partial T^1(\nu,\cos\theta)}{\partial\cos\theta}\bigg|_{1+2/\nu}\right] - \frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^5}\ .$$

APPENDIX B

From threshold to 0.6 GeV, we use phase shifts based on a comparison between K_{e_4} data and a rigorous representation for $\pi\pi$ amplitudes.[16] More specifically, we assume that

$$Q\cot\delta_0^0 = \frac{16.4}{s-0.05} - 0.36\ , \qquad \text{(B1)}$$

$$Q\cot\delta_0^2 = \frac{-45.8}{s-2.04} - 0.97\ , \qquad \text{(B2)}$$

$$Q\cot\delta_1^1 = \frac{97.7}{s-4} - 2.79 - 0.0262s\ . \qquad \text{(B3)}$$

where

$$Q \equiv (1-4/s)^{1/2}\ .$$

Equation (B1) corresponds to a scattering length $a_0 = 0.26$, with $\delta_0^0 = 43°$ at 0.5 GeV. Equation (B2) corresponds to $a_2 = -0.041$, with $\delta_0^2 = -9°$ at 0.5 GeV. Equation (B3) corresponds to $a_1 = 0.040$, with $m_\rho = 770$ MeV, and $\Gamma_\rho = 146$ MeV.[16] We neglect absorptive parts with $l \geq 2$ below 0.6 GeV.

Between 0.6 and 1.9 GeV, we use the $I=0$ and $I=1$ phase shifts and elasticities of Hyams *et al.*,[17] and the $I=2$ phase shifts and elasticities of Durusoy *et al.*[18] (The ρ resonance of Ref. 17 has a mass of 778 MeV and a width of 152 MeV.) We neglect absorptive parts with $l \geq 4$ below 1.9 GeV.

Above 1.9 GeV, we use Regge theory, and assume that

$$\mathrm{Im}T^0 = \gamma_P(t)(s/\bar{s})^{\alpha_P(t)} + \gamma_f(t)(s/\bar{s})^{\alpha_f(t)}\ ,$$

$$\mathrm{Im}T^1 = \gamma_\rho(t)(s/\bar{s})^{\alpha_\rho(t)}\ ,$$

$$\mathrm{Im}T^2 = 0,$$

where

$$t \equiv -\tfrac{1}{2}(s-4)(1-\cos\theta)\ ,$$

and the subscripts P, f, and ρ denote Pomeron, f_0, and ρ exchange, respectively. The scale factor $\bar{s}$ is chosen to be 1 GeV^2.

We need $\mathrm{Im}T^0$ only as a contribution to $\mathrm{Im}A^1$ in the forward direction ($t=0$). An asymptotic total cross section of 17 mb (Ref. 19) is incorporated by assuming that

$$\gamma_P(0) = 1.34\ ,$$

$$\alpha_P(0) = 1\ .$$

We use ρ-f_0 exchange degeneracy to write

$$\alpha_f(t) = \alpha_\rho(t)\ ,$$

$$\gamma_f(t) = \tfrac{3}{2}\gamma_\rho(t)\ .$$

Assuming that

$$\alpha_\rho(t) = 0.50 + 0.90(t/\bar{s})\ ,$$

a study of $\pi\pi$ charge-exchange data above 1 GeV indicates that[4]

$$\gamma_\rho(t) \cong 0.67 + 1.78(t/\bar{s}) + 0.41(t/\bar{s})^2 - 0.17(t/\bar{s})^3$$

for $-1.0\ \mathrm{GeV}^2 < t < 0.1\ \mathrm{GeV}^2$.

We also need the value of $\mathrm{Im}A^{(1)1}$ above 1.9 GeV. In the case of total absorption ($\eta_1^1 = 0$), $\mathrm{Im}A^{(1)1}$ would equal 0.5. Our high-energy integrals begin at 1.9 GeV, however, where absorption is quite incomplete.[17] We therefore assume the effective value

$$\mathrm{Im}A^{(1)1} = 0.25$$

above 1.9 GeV.

The preceding absorptive parts represent our favored values. Uncertainties in the S-wave contributions to the sum rules (7) may be generously estimated at 30%. The P-wave contributions are strongly dominated by the ρ resonance, whose width is uncertain by about 10%.[20] The D_0 contributions are dominated by the f_0 resonance, whose $\pi\pi$ partial width is uncertain by about 20%.[20] The D_2 contributions are negligible, so an estimate of their uncertainties would be superfluous. The F-wave contributions are dominated by the g resonance, whose $\pi\pi$ partial width is uncertain by about 25%.[20]

For the high-energy contributions, we note that $\gamma_P(0)$ is uncertain by about 20%.[19] Pomeron exchange may not be wholly effective at an energy as low as 1.9 GeV, however, so we estimate the contributions of $\mathrm{Im}T^0_{\mathrm{HE}}$ to be uncertain by 30%.

Reggeized ρ exchange appears to be effective (via duality) above 1 GeV,[4] so we estimate the contributions of $\mathrm{Im}\,T^1_{\mathrm{HE}}$ to have the same uncertainty as γ_ρ, namely 15%.[4] Finally, we estimate the uncertainty in contributions of $\mathrm{Im}A^{(1)1}_{\mathrm{HE}}$ to be 50%.

The preceding uncertainties are statistically uncorrelated, so by a routine analysis we estimate the uncertainties in the four "total" values in the bottom row of Table I to be 29%, 33%, 18%, and 15%, respectively.

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[1]E. P. Tryon, Phys. Lett. 38B, 527 (1972).

[2]James Dilley, Phys. Rev. D 8, 905 (1973).

[3]Cf. J. S. Kang and H. J. Schnitzer, Phys. Rev. D 12, 841 (1975); E. P. Tryon, Phys. Rev. Lett. 36, 455 (1976).

[4]E. P. Tryon, Phys. Rev. D 11, 698 (1975).

[5]Cf. M. G. Olsson, Phys. Rev. 162, 1338 (1967).

[6]We assume the domain of convergence to be governed by the boundaries of the double spectral functions; Cf. G. F. Chew and S. Mandelstam, Nuovo Cimento 19, 752 (1961), and references cited therein.

[7]Consider a model which is analytic and crossing-symmetric *by construction*, wherein $\mathrm{Im}A^{(l)I}(\nu)=0$ for all $l>L$, $\nu>0$, $I=0$, 1, and 2. For $l\leq L$, $\nu>0$, the $\mathrm{Im}A^{(l)I}$ are regarded as arbitrary in this model. Equation (6) is valid over the entire left cut, because the Legendre series is truncated at L and therefore converges everywhere. Since analyticity is assumed, all the usual dispersion relations are satisfied, provided that enough subtractions are performed to obtain convergence of the integrals. [Crossing symmetry imposes constraints on many of the subtraction constants, as well as determining the left cuts by equations similar to (6). No other type of constraint is implied by crossing symmetry in this model, provided the amplitudes are *defined* by dispersion relations.] By inspection of Eqs. (2) and (6), the $(n-1)$th derivative of $A^{(l)1}$ at $\nu=0$ satisfies an *unsubtracted* dispersion relation for all $n>(L+1)$. This derivative is determined by the nth negative moment of $\mathrm{Im}A^{(l)1}$. Since the $\mathrm{Im}A^{(l)1}$ with $l\leq L$ are *arbitrary* for $\nu>0$ in this model, each must make *identical* contributions to the left- and right-hand sides of Eq. (5), or else it would be violated, Q.E.D.

[8]Cf. B. Hyams *et al.*, Nucl. Phys. B64, 134 (1973). The centrifugal barrier, combined with unitarity, implies that $\mathrm{Im}A^{(l)I}$ vanishes at threshold like $\nu^{(2l+1/2)}$.

[9]For $n\geq 4$, evaluation of the right-hand side of Eq. (7) requires a knowledge of $\mathrm{Im}\,T^1$ for $\cos\theta$ within a neighborhood of $(1+2/\nu)$. This is sufficiently near the physical region for the Legendre series to converge, however, so that knowledge of $\mathrm{Im}\,T^1$ within the physical region provides enough information to make a simple and rigorous extrapolation (cf. Ref. 6).

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[12]E. P. Tryon, Phys. Rev. D 5, 1039 (1972).

[13]L. A. P. Balazs, Phys. Rev. 132, 867 (1963), and references cited therein.

[14]The mutual normalization of N and D was chosen such that $D(0)=1$.

[15]T. Gibbons and J. Dilley, Phys. Rev. D 3, 1196 (1971).

[16]E. P. Tryon, Phys. Rev. D 10, 1595 (1974).

[17]B. Hyams *et al.*, Ref. 8. We use the energy-dependent amplitudes. The reader is cautioned that a typesetting error is present in Eq. (13b) of this work, where $\gamma_\pi{}^2$ is dimensionally incorrect and does not reproduce the phase shift δ^0_2 of Fig. 5.

[18]N. B. Durusoy *et al.*, Phys. Lett. 45B, 517 (1973).

[19]W. J. Robertson and W. D. Walker, Phys. Rev. D 7, 2554 (1973), and references cited therein.

[20]Particle Data Group, Phys. Lett. 50B, 1 (1974).